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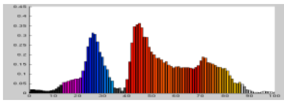


Signals and Systems
Lecture 9

Outline

- **System properties in terms of impulse response.**
- **Impulse Response, Step Response and LTI Continuous-time system properties.**
- **Systems with Finite-duration and Infinite-duration Impulse Response FIR and IIR**

System properties in terms of impulse response

Since an LTI system is completely characterized by its impulse response, we can specify system properties in terms of impulse response and determine whether or not an LTI system is memoryless, causal and stable.

1. Memoryless system

In general case, for an LTI system a memoryless system is one for which $y[n]=kx[n]$, where k is a constant called the **Gain constant**. In the case where the LTI system is described by discrete convolution, the impulse response can be written as $h[n]=k\delta[n]$, recall that $\delta[n]$ is zero except at the origin, we see that an LTI system is memoryless if and only if $h[n]=0$, where $n\neq 0$.

2. Causality for LTI system

Thus impulse response $h[n]$ for a causal LTI system must satisfy the condition

$$h[n]=0 \text{ for } n<0.$$

Looking at $y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$, we see that for causal system we can write the output signal as:

$$y[n]=\sum_{k=0}^{\infty}h[k]x[n-k]$$

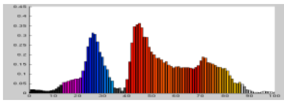
Causal signal is one for which $x[n]=0$ for $n<0$
Anticausal signal is one for which $x[n]=0$ for $n>0$

To proof the condition of convolution-based LTI system causality, let consider an LTI system having an output at time n_0, we can write the output of the system as summation of two sets as

$$y[n_0]=\sum_{k=-\infty}^{\infty}h[k]x[n_0-k]=\sum_{k=0}^{\infty}h[k]x[n_0-k]+\sum_{k=-\infty}^{-1}h[k]x[n_0-k]=$$
$$=\left[h[0]x[n_0]+h[1]x[n_0-1]+h[2]x[n_0-2]+...\right]+$$
$$+\left[h[-1]x[n_0+1]+h[-2]x[n_0+2]+h[-3]x[n_0+3]...\right]$$

$\left[h[0]x[n_0]+h[1]x[n_0-1]+h[2]x[n_0-2]+...\right]$-this set is the **present and the past** values of the input signal.

$\left[h[-1]x[n_0+1]+h[-2]x[n_0+2]+h[-3]x[n_0+3]...\right]$-this set is the **future values** of the input signal.



Know, if the output of a causal system depends only on present and past-values of the input, then clearly the impulse response must satisfy the condition $h[n]=0$ for $n<0$.

3. Stability for LTI system

A system is stable if every bounded input produces a bounded output. Consider input $x[n]$ such that $|x[n]|<B$ for all n. To investigate the stability of LTI system described by its impulse response, taking the absolute value of both sides, we can write the following:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

From triangle inequality for complex numbers $|z_1+z_2| \le |z_1| + |z_2|$ (the absolute value of the sum of terms is always less than or equal to the sum of the absolute values of the terms). Hence

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

If the input is bounded, $|x[n]|<B$, then

$$|y[n]| \le B \sum_{k=-\infty}^{\infty} |h[k]|$$

From this equation, the output is bounded if the impulse response of the system satisfies the condition:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

LTI system is stable if its impulse response is absolutely summable

Examples:

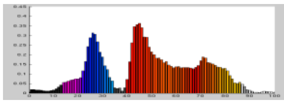
a Suppose that the impulse response for a discrete LTI system is $h[n]=5nu[n-1]$, is the system memoryless? Is it causal? Is it BIBO stable?

- The system is **not memoryless:** since there are terms for which $h[n]\neq 0$, when $n \neq 0$.
- The system is **causal system:** since $h[n]=0$ for $n<0$.
- The system is not **BIBO stable**, because if we compute the sum of $|h[n]|$, we have

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |5ku[k-1]| = \sum_{k=1}^{\infty} |5k| \to \infty \text{ (not finite)}$$

b Discuss the memoryless, causality and stability of the system with impulse response: $h[n]=5\sin(\frac{3}{2}\pi n)u[n+5]$

- It might be helpful to plot the function using matlab code.
- The system is **not memoryless:** since there are terms for which $h[n]\neq 0$, when $n \neq 0$.
- The system is **not causal system:** since $h[n]\neq 0$ for $n<0$.



- The system is not **BIBO stable**, because if we compute the sum of $|h[n]|$, we have

$$\sum_{k=-\infty}^{\infty}|h[k]| = \sum_{k=-\infty}^{\infty}\left|5\sin(\frac{3}{2}\pi n)u[n+5]\right| = \sum_{k=-5}^{\infty}\left|5\sin(\frac{3}{2}\pi n)\right| \to \infty \text{ (not finite)}$$

Impulse Response, Step Response and LTI Continuous-time system properties

❖ The **signal** can be represented during $\delta(t)$ as the following:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

❖ The **output signal** can be represented as

$$y(t) = \hat{T}\{x(t)\} = \hat{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} = \int_{-\infty}^{\infty} x(\tau)\hat{T}\{\delta(t-\tau)\}d\tau$$

❖ From time-shifting property $h(t-\tau) = \hat{T}\{\delta(t-\tau)\}$, we can obtain the **main formula** for the convolution integral of the **impulse response** of the continuous-time system and any input :

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

❖ **Step response** of a C-T system can be expressed as

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

❖ Form commutative property

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

Note that if we know the step response of a system, we can find the impulse response by computing

$$h(t) = \frac{ds}{dt}$$

❖ **LTI Continuous-time system properties:**

1. Memoryless

$$y(t) = k \cdot x(t) \Rightarrow h(t) = k \cdot \delta(t) \Rightarrow h(t) = 0, \text{ when } t \neq 0$$

So, A linear time-invariant system is **memoryless**, if the impulse response of the system satisfies the condition

$$h(t) = 0, \text{ when } t \neq 0$$

2. Causality

A linear time-invariant system that is **causal** is one for which

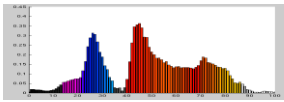
$$h(t) = 0 \text{ when } t < 0$$

3. Stability

If the impulse response of a linear time-invariant system satisfies

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

we say that the system is stable.



Examples:

a) For an LTI continuous system with the unit impulse response $h(t) = e^{-2t}u(t)$, determine the response $y(t)$ for the input $x(t) = e^{-t}u(t)$.

Solution:

Here both $x(t)$ and $h(t)$ are causal, from the convolution integral equation, we obtain

$$\int_0^t x(\tau)h(t-\tau)d\tau \qquad t \geq 0$$

Because $x(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$ then $x(\tau) = e^{-\tau}u(\tau)$ and $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$

Remember that the integration is performed with respect to τ (not t), and the region of integration is $0 \leq \tau \leq t$. Hence, $\tau \geq 0$ and $t - \tau \geq 0$. Therefore, $u(\tau) = 1$ and $u(t-\tau) = 1$; consequently

$$y(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau \qquad t \geq 0$$

Because this integration is with respect to τ, we can pull e^{-2t} outside the integral, giving us

$$y(t) = e^{-2t}\int_0^t e^{\tau} d\tau = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t} \qquad t \geq 0$$

Moreover, $y(t) = 0$ when $t < 0$. Therefore

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

b) The linear time-invariant system with $h(t) = 4e^{-2t}u(t)$

The system is stable, causal, but has memory

c) Consider an impulse response given by $h(t) = 5\delta(t)$. Is this system memoryless, causal, and stable?

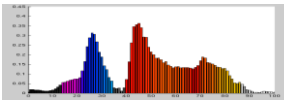
Solution:

- $h(t) = 0,$ when $t \neq 0 \Rightarrow$ The system is memoryless, it's immediately follows that the system is causal.
- To determine the stability, notice that , $\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} 5 \cdot \delta(t) d\tau = 5\int_{-\infty}^{\infty} \delta(t) d\tau = 5 < \infty$, therefore the system is stable.

d) An LTI system has the impulse response function $h(t) = e^{-2t}u(t)$. Is the system memoryless? Is it causal? Is it stable?

Solution

- The system is not memoryless, since $h(t)$ is not zero when $t \neq 0$.
- The system is causal, since $h(t) = 0$ for $t < 0$.
- The system is stable, since $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-t}u(t) dt = \int_0^{\infty} e^{-t} dt = -e^{-t}\Big|_0^{-\infty} = 1$.



Systems with Finite-duration and Infinite-duration Impulse Response FIR and IIR

FIR Systems has an impulse response that is zero outside of some **finite** time interval (for example **causal FIR Systems**):

$$h[n]=0, n<0 \text{ and } n\geq M$$

The convolution formula for such a system reduces to

$$y[n]=\sum_{k=0}^{M-1} h[k]x[n-k]$$

FIR System has a finite memory of length equal to M samples.

IIR Systems has an **infinite-duration** impulse response. It's output based on convolution formula

$$y[n]=\sum_{k=0}^{\infty} h[k]x[n-k] \text{ (Taking into account the causality)}$$

IIR System has an infinite memory of length.